Uncertainty relations

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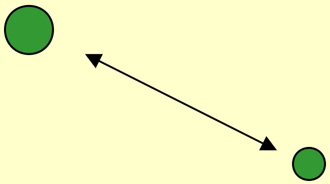
-- Introduction

-- Heisenberg-Kennard-Robertson

-- Precision-disturbance interpretastion

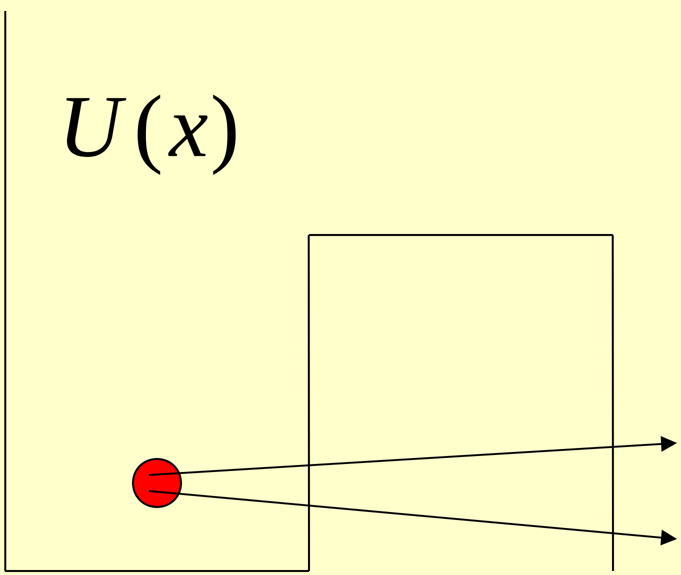
-- Joint measurement interpertation

$$\Delta x \, \Delta p \geq \hbar$$



why quantum particles do not fall into each other

$$\Delta x \to 0 \;\Rightarrow\; \frac{\Delta p^2}{2m} \to \infty$$



tunneling

$$\frac{p^2}{2m} = E - U(x) < 0$$

Quantum mechanics = waves + probability theory

state: vector in complex linear space

$$|\psi\rangle = (\psi_1, \ldots, \psi_N)^T, \ \langle\psi| = (\psi_1 *, \ldots, \psi_N *) \qquad \langle\psi|\psi\rangle = \sum_{k=1}^{N} \psi_k \psi_k *$$

observable: linear, hermitean operator

$$A = \sum_{k=1}^{N} a_k |a_k\rangle\langle a_k|, \quad A|\psi\rangle = \sum_{k=1}^{N} a_k |a_k\rangle\langle a_k|\psi\rangle \qquad \langle a_k|a_l\rangle = \delta_{kl}$$

$$|\langle a_k|\psi\rangle|^2$$ ← **probability**

Standard uncertainty relations

$$\langle\psi|(\alpha A+\beta B)^2|\psi\rangle=\alpha^2\langle A^2\rangle+\beta^2\langle B^2\rangle+\alpha\beta(\langle AB\rangle+\langle BA\rangle)$$

$$\langle A^2\rangle\langle B^2\rangle\geq\langle AB\rangle\langle BA\rangle=|\langle AB\rangle|^2$$

Cauchy-Schwartz inequality

$$\langle AB\rangle=\frac{1}{2}(\langle AB\rangle+\langle BA\rangle)+\frac{1}{2}(\langle AB\rangle-\langle BA\rangle)$$

$$\langle A^2\rangle\langle B^2\rangle\geq\frac{1}{4}(\langle AB\rangle+\langle BA\rangle)^2+\frac{1}{4}|\langle[A,B]\rangle|^2$$

$$A=Q-\langle Q\rangle,\quad B=P-\langle P\rangle$$

Kennard-Robertson uncertainty relation (1927)

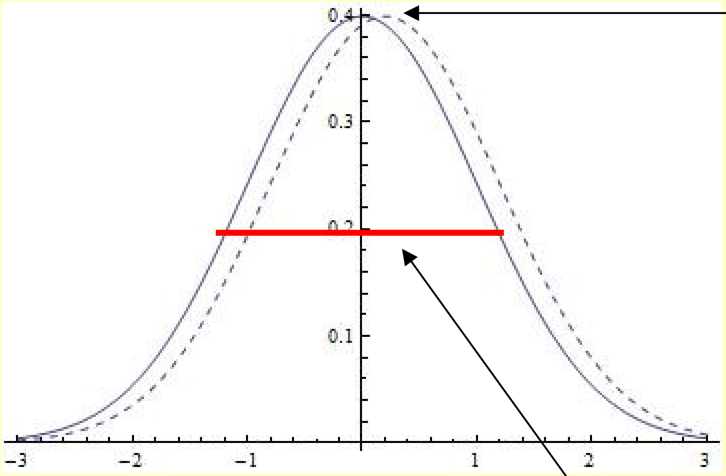
$$\langle (Q - \langle Q \rangle)^2 \rangle \; \langle (P - \langle P \rangle)^2 \rangle \geq \frac{\hbar^2}{4}, \quad [Q, P] = i\hbar$$

impossible to prepare a quantum state with precise values of .....

Illustration for atoms in crystalls: *W Jauch*, *Am J Phys 1993*.

MnF_2

F	$\sigma(x) \geq \sigma_{\min}(x)$
$295\,K$	$11.78 \geq 4.47 \times 10^{-12}\ m$
$60\ \ K$	$7.10 \geq 6.55$
$15\ \ K$	$6.71 \geq 6.60$



$\approx 10^{-14}\ m$

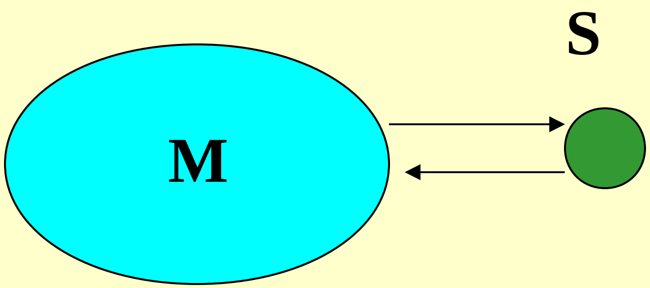
$\sigma(x) \approx 10^{-12}\ m$

Bohr-Heisenberg

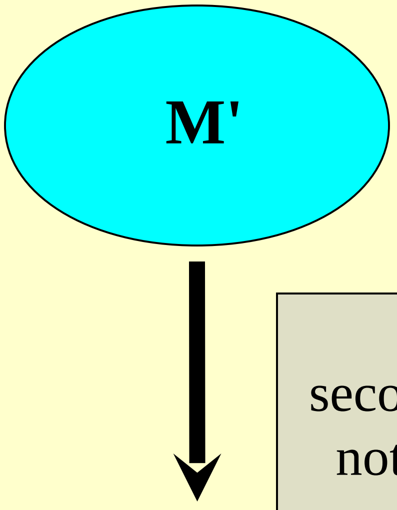
under <u>joint</u> measurements the precisions hold .....?

measuring one variable perturbs another ?
error-disturbance

Measurement



first stage: interaction between 2 quantum systems
correlations between **M** and **S**



second stage: appearance of definite measurement results events
not decribed by quantum mechanic without additional axioms

measurement problem, unconventional quantum theories etc

Error-disturbance relation between conjugate variables

$$\begin{pmatrix} Q_M(\tau) \\ Q_S(\tau) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} Q_M(0) \\ Q_S(0) \end{pmatrix}$$

linear transformation

$$\begin{pmatrix} P_M(\tau) \\ P_S(\tau) \end{pmatrix} = \frac{1}{ad-bc} \begin{pmatrix} -c & d \\ a & -b \end{pmatrix} \begin{pmatrix} P_M(0) \\ P_S(0) \end{pmatrix}$$

requires special Hamiltonian

$$Q_M(\tau) - Q_S(0)$$

error in measuring the coordinate

$$P_S(\tau) - P_S(0)$$

disturbance of the momentum

$$a = 0, b = 1 : \quad Q_M(\tau) = Q_S(0)$$

perfect measurement with finite disturbance

Simultaneous measurement of two non-commuting variables

$$|\Phi\rangle = |\psi_S\rangle|\psi_M\rangle$$

initial time

$$N_P = P_M(\tau) - P_S(0), \quad N_Q = \Pi_M(\tau) - Q_S(0)$$

uncertainties

$$\langle\Phi|N_P|\Phi\rangle = \langle N_P\rangle = \langle N_Q\rangle = 0 \quad \text{for any } |\psi_S\rangle$$

unbiased

$$[P_M(\tau), \Pi_M(\tau)] = 0$$

non-commuting variables are mapped to commuting ones

$$\langle [N_P, N_Q] \rangle = \langle [Q_S(0), P_S(0)] \rangle$$

$$\langle N_P^2 \rangle \langle N_Q^2 \rangle \geq \frac{1}{4} |\langle [Q_S(0), P_S(0)] \rangle|^2$$

irreducible noise

$$\langle (P_M(\tau) - \langle P_M(\tau) \rangle)^2 \rangle \, \langle (\Pi_M(\tau) - \langle \Pi_M(\tau) \rangle)^2 \rangle \geq |\langle [Q_S(0), P_S(0)] \rangle|^2$$

4 times larger

Final state of commuting variables is noisy

Arthurs & Goodman 1988

Conclusion

- Uncertainty relation as an intrinsic feature of quantum states: clear.
- As a feature of joint measurements: more is to be done (unbiasedness?)
- As error-disturbance relation: does not hold in its literal form. Error-disturbance is a more complex issue.

Quantum mechanics

$|\psi\rangle$

state: vector in complex linear space

$$A = A^\dagger : \; A = \sum_k a_k |a_k\rangle\langle a_k|$$

observable: linear, hermitean operator

$$A(t) = U_t^\dagger A U_t, \quad U = e^{-itH/\hbar}, \quad UU^\dagger = 1$$

Heisenberg dynamics

Hamiltonian

$$|\psi(t)\rangle = U_t|\psi(0)\rangle$$

Schroedinger dynamics

$$\langle A\rangle_t = \langle\psi(t)|A|\psi(t)\rangle = \langle\psi(0)|A(t)|\psi(0)\rangle$$

average

Warning: non-commuting operators can be sometimes measured jointly

$$[A, B] \neq 0, \quad \text{can exist } |\psi'\rangle \ : \quad A|\psi'\rangle = a|\psi'\rangle, \ B|\psi'\rangle = b|\psi'\rangle.$$